University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO EXAMINATION 4

Directions. Do both problems (weights are indicated). This is a closed-book closed-note exam except for four $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he will not give hints and will be obliged to write your question and its answer on the board. Roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (50 points)
Four thin rods of length a and mass m are welded together to form a square picture frame of side a and mass $4m$. In a set of body axes tied to the frame, the frame lies in the xy plane; its corner is at the origin and its sides lie along the $\hat{x}$ and $\hat{y}$ axes. (Note that the origin does not coincide with the frame's CM.)

The frame is rotating counterclockwise about the (body or space) $\hat{y}$ axis with uniform angular velocity ω. In the body axes, what are the components of its angular momentum?

Solution:
The angular momentum is $\tilde{L} = \mathcal{I}\tilde{\omega}$. Displaying the components,

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} \mathcal{I}_{xx} & \mathcal{I}_{xy} & \mathcal{I}_{xz} \\ \mathcal{I}_{yx} & \mathcal{I}_{yy} & \mathcal{I}_{yz} \\ \mathcal{I}_{zx} & \mathcal{I}_{zy} & \mathcal{I}_{zz} \end{pmatrix} \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} = \begin{pmatrix} \mathcal{I}_{xy}\omega \\ \mathcal{I}_{yy}\omega \\ \mathcal{I}_{zy}\omega \end{pmatrix} .$$

$\mathcal{I}_{zy}$ vanishes because the frame is in the plane $z = 0$, so we are left with the task of calculating $\mathcal{I}_{xy}$ and $\mathcal{I}_{yy}$. Orient the picture frame so that "up" is along $\hat{y}$ and "to the right" is along $\hat{x}$, and denote the rods by "up", "down", "left", and "right". The diagonal component $\mathcal{I}_{yy}$ is the frame's scalar moment of inertia for rotation about the $\hat{y}$ axis. To it the right-hand rod contributes ma^2 and the left-hand rod contributes 0 (because it coincides with the $\hat{y}$ axis). The top and bottom rods each contribute

$$\frac{m}{a}\int_0^a (x^2 + z^2)\, dx = \tfrac{1}{3}ma^2 .$$

The total is

$$I_{yy} = ma^2 + 2(\tfrac{1}{3}ma^2) = \tfrac{5}{3}ma^2 .$$

As for the off-diagonal component $\mathcal{I}_{xy}$, it is proportional to xy integrated over the frame. For the left-hand rod $x = 0$ and for the bottom rod $y = 0$, so they don't contribute. The contribution of the top rod is

$$-\frac{m}{a}\int_0^a xy\, dx = -\tfrac{1}{2}ma^2 ;$$

the right-hand rod's contribution is the same. Then

$$I_{xy} = -\tfrac{1}{2}ma^2 - \tfrac{1}{2}ma^2 = -ma^2 .$$

Putting it together,

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = ma^2\omega \begin{pmatrix} -1 \\ \frac{5}{3} \\ 0 \end{pmatrix} .$$

2. (50 points)
Consider a football in a force-free environment. Neglecting its seams and laces, approximate it as cylindrically symmetric about its own $\hat{3}$ (body) axis. At $t = 0$, a set of space ($'$) axes is (momentarily) coincident with the body axes. As seen in the space axes at $t = 0$, the CM of the football is at rest at the origin; the football is rotating counterclockwise about the (space) $\hat{2}'$ axis with angular velocity ω (this is "end-over-end" rotation).

Give a qualitative description of the football's motion for $t > 0$ as seen in its own (body) system. Justify your assertions. Does the football's angular momentum appear to stay constant when it is observed in the body system?

Solution:

You may use its cylindrical symmetry to argue that the football's $\hat{2}$ axis a principal axis. Pure rotation about any principal axis always proceeds without wobbling, so $\vec{\omega}$ and $\mathbf{L}$ are constant (and nonzero!). This simple argument is sufficient to earn full credit.

Alternatively, you may use the Euler equations. In the body system,

$$\begin{aligned}
\mathcal{I}_{33}\dot{\omega}_3 - (\mathcal{I}_{11} - \mathcal{I}_{22})\omega_1\omega_2 &= N_3 \\
\mathcal{I}_{11}\dot{\omega}_1 - (\mathcal{I}_{22} - \mathcal{I}_{33})\omega_2\omega_3 &= N_1 \\
\mathcal{I}_{22}\dot{\omega}_2 - (\mathcal{I}_{33} - \mathcal{I}_{11})\omega_3\omega_1 &= N_2 .
\end{aligned}$$

In this force-free environment, all torque components N_i vanish. Since ω_2 is the only nonvanishing component of $\vec{\omega}$, all of the terms proportional to $\omega_i\omega_j$ also vanish. Therefore the time derivatives of all the components of $\vec{\omega}$ vanish, forcing $\vec{\omega}$ to remain constant. The angular momentum $\mathbf{L} = \hat{2}\mathcal{I}_{22}\omega$ likewise remains constant. Again this argument is sufficient for full credit.

Optionally, you may be worried about the stability of rotation about the $\hat{2}$ axis. According to lecture notes (**10.3**),

$$\frac{\ddot{\omega}_i}{\omega_i} \propto (I_k - I_j)(I_k - I_i) \; ,$$

where $\hat{k}$ is the main axis of rotation and $\hat{\imath}$ is an orthogonal direction along which a tiny stray component of $\vec{\omega}$ might grow. However, for the cylindrically symmetric football, two of the three principal moments of inertia are equal, so the right-hand term must vanish; correspondingly $\ddot{\omega}_i$ vanishes and the end-over-end rotation is stable. (You lose credit if you claim that this rotation is unstable, but you are not required to show that it is stable.)